Pergamon

PII: S0021-8928(98)00124-5

J. Appl. Maths Mechs, Vol. 62, No. 6, pp. 971–974, 1998
© 1999 Elsevier Science Ltd
All rights reserved. Printed in Great Britain
0021-8928/98/\$—see front matter

THE RELATIVE EQUILIBRIA OF A TETHERED GYROSTAT IN A CENTRAL NEWTONIAN FIELD†

A. A. BUROV and H. TROGER

Moscow

(*Received 5 January* 1998)

The motion of an orbital tether system comprising a massive body and a gyrostat of small mass attached to it by a non-extensible weightless tether is examined. The body performs unperturbed motion in a Kepler orbit. There are several different equilibria of the system relative to a uniformly rotating system of coordinates. These equilibria are interpreted geometrically using Mohr circles. Despite being the simplest example of an orbital tether system with a gyrostat, it exhibits a wealth of dynamic properties. There are also more complex orbital tether systems which contain more than one gyrostat [1]. © 1999 Elsevier Science Ltd. All rights reserved.

Let one end of a tether (point A) be attached to a massive body that performs uniform motion in a circular Kepler orbit, which cannot be disturbed by the motion of a second body. If N is an attracting centre, then $\mathbf{R} = NA = R\gamma$, where R is the radius of the orbit and γ is a unit vector. The other end of the tether (point B) is attached to the gyrostat frame, and the points A and B are a distance l apart. Then $\mathbf{L} = AB = l\mathbf{h}$, where $\mathbf{h}$ is also a unit vector. The point C is taken as the centre of mass of the gyrostat, and the vector $\mathbf{k} = BC$ is assumed to be fixed in its casing. The vector $\mathbf{r} = NC$, which defines the position of the centre of mass of the gyrostat in absolute space, is related to the above vectors by the equation

$$\mathbf{r} = \mathbf{R} + \mathbf{L} + \mathbf{k}$$

Suppose that inside the gyrostat there is a rotor which rotates relative to its casing with constant relative angular velocity $\dot{\varphi}$. This is an example of a Kelvin gyrostat (see [2, pp. 256–19] for example). Let $NX_1X_2X_3$ be an orbital system of coordinates which rotates uniformly about the axis NX_2, formed by the unit vector γ, β and $\boldsymbol{\alpha} = \boldsymbol{\beta} \times \boldsymbol{\gamma}$, where β and α are the normal to the orbital plane and the tangent to the circular orbit at the point A. In the rotating system of coordinates $NX_1 \parallel \boldsymbol{\alpha}$, $NX_2 \parallel \boldsymbol{\beta}$, $NX_3 \parallel \boldsymbol{\gamma}$. We also introduce a system of coordinates which is fixed in the gyrostat frame, the axes coinciding with the principal axes of the central tensor of inertia $\mathbf{I}$ of the gyrostat as a whole. We will denote this system of coordinates of $Cx_1x_2x_3$ (Fig. 1). Let $\omega = (\omega_1, \omega_2, \omega_3)$ be the vector of the absolute angular velocity of the gyrostat and let $\mathbf{K} = (K_1, K_2, K_3)$ be the kinetic moment of the rotor (KMR). The magnitude of the vector $\mathbf{K}$ is proportional to the magnitude of the angular velocity of the rotor. Here and below, unless otherwise stated, all vector magnitudes are given as their projections onto axes associated with the gyrostat.

We will use the Routh–Lyapunov method to find the relative equilibria in the orbital system of coordinates and investigate their stability properties. This involves investigating the critical points of changed potential, considered as a function at the common level of the "geometric" integrals

$$\pi_\gamma = (\boldsymbol{\gamma}, \boldsymbol{\gamma}) - 1 = 0,\ \ \pi_{\gamma\beta} = (\boldsymbol{\gamma}, \boldsymbol{\beta}) = 0,\ \pi_\beta = (\mathbf{b}, \boldsymbol{\beta}) - 1 = 0,\ \ \pi_\mathbf{h} = (\mathbf{h}, \mathbf{h}) - 1 = 0$$

which express the fact that the system of vectors $(\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma})$ is orthonormalized and $\mathbf{h}$ is a unit vector. In order to use the method of Lagrange multipliers, we construct the Routh function ($\dot{\psi}$ is the constant orbital angular velocity)

$$W_\pi = \frac{W_K}{\dot{\psi}^2} + 3\lambda\pi_{\gamma\beta} + \frac{\nu}{2}\pi_\beta - \tfrac{3}{2}\sigma\pi_\gamma + \frac{\chi}{2}ml\pi_\mathbf{h}$$

where

$$W_K \equiv -\tfrac{1}{2}\dot{\psi}^2[(\mathbf{I}\boldsymbol{\beta},\ \boldsymbol{\beta}) - m(\boldsymbol{\beta},\ l\mathbf{h}+\mathbf{k})^2] - \dot{\psi}(\mathbf{K},\ \boldsymbol{\beta}) + \tfrac{3}{2}\dot{\psi}^2[(\mathbf{I}\boldsymbol{\gamma},\ \boldsymbol{\gamma}) - m(\boldsymbol{\gamma}, l\mathbf{h}+\mathbf{k})^2] + \mathrm{const}_K$$

is the changed potential [3].

The critical points are found from a system of 13 algebraic equations, in the general case non-linear, of the form

†*Prikl. Mat. Mekh.* Vol. 62, No. 6, pp. 1049–1052, 1998.

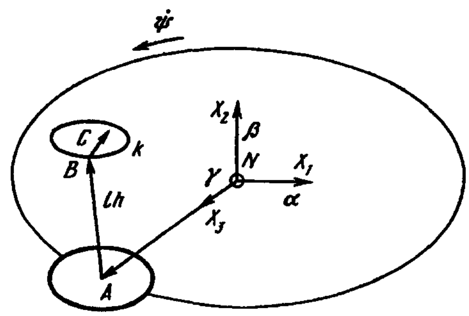

Fig. 1.

$$\frac{\partial W_\pi}{\partial \gamma}=3[(\mathbf{I}(\mathbf{h})-\sigma E)\gamma+\lambda\beta]=3[(\mathbf{I}-\sigma E)\gamma+\lambda\beta-m(\gamma,\ l\mathbf{h}+\mathbf{k})(l\mathbf{h}+\mathbf{k})]=0$$

$$\frac{\partial W_\pi}{\partial \beta}=3\lambda\gamma+(\nu E-\mathbf{I}(\mathbf{h}))\beta-\dot\psi^{-1}\mathbf{K}=3\lambda\gamma+(\nu E-\mathbf{I})\beta-\dot\psi^{-1}\mathbf{K}+m(\beta,\ l\mathbf{h}+\mathbf{k})(l\mathbf{h}+\mathbf{k})=0$$

$$\frac{\partial W_\pi}{\partial \mathbf{h}}=ml[(\beta,\ l\mathbf{h}+\mathbf{k})\beta-3(\gamma,\ l\mathbf{h}+\mathbf{k})\gamma+\chi\mathbf{h}]=0$$

$$\frac{\partial W_\pi}{\partial \lambda}=\frac{\partial W_\pi}{\partial \nu}=\frac{\partial W_\pi}{\partial \sigma}=\frac{\partial W_\pi}{\partial \chi}=0$$

Here E is the 3×3 identity matrix.

It turns out that there are particular solutions for which the four points N, A, B and C lie on a straight line. If

$$i_{h\gamma}=(\mathbf{h},\ \gamma),\ \ i_{k\gamma}=(\mathbf{k},\ \gamma),\ \ \sigma^*=\sigma+m(li_{h\gamma}+i_{k\gamma})^2$$

the gyrostat orientation is given by the equations

$$(\mathbf{I}-\sigma^* E)\gamma+\lambda\beta=0,\quad \dot\psi^{-1}\mathbf{K}=3\lambda\gamma+(\nu E-\mathbf{I})\beta \tag{1}$$

Apart from the notation, Eqs (1) are the same as the equations of the relative equilibria of a satellite-gyrostat. The position of the vector $\mathbf{k}$ in the gyrostat is specified. Then for the given motions, the direction of the vector γ is also specified. Thus it is sensible to use what is known as a semi-inverse method, in which the orientation of the gyrostat in relation to the vector γ is specified, and the orientation of the gyrostat in relation to the vectors α and β is investigated as a function of the KMR $\mathbf{K}$, which ensures that orientation. As in [4] we have

$$\sigma^*=(\mathbf{I}\gamma,\ \gamma),\ \ \lambda^2=(\mathbf{I}\gamma\times\gamma,\ \mathbf{I}\gamma\times\gamma),\ \ \beta=-\lambda^{-1}(\mathbf{I}-\sigma^* E)\gamma,\ \ \alpha=\beta\times\gamma$$

The KMR which ensures that orientation is found from the relations

$$\dot\psi^{-1}\mathbf{K}=3\lambda\gamma+(\nu E-\mathbf{I})\beta \tag{2}$$

If $\lambda = 0$, that is, γ is an eigenvector of the matrix $\mathbf{I}$, the gyrostat can have any orientation relative to β, and this can be controlled by the KMR in accordance with (2). A KMR which provides a given gyrostat orientation forms a one-parameter family relative to the parameter ν, which can be written as $\nu = (\mathbf{I}\beta, \beta) + \dot\psi^{-1}(\mathbf{K}, \beta)$.

Unlike σ^*, the parameters λ and ν have quite a simple mechanical interpretation. The parameter λ is proportional to the magnitude of the moment of forces of Newtonian attraction. The parameter ν, as in the case of a satellite-gyrostat [4], is proportional to the projection of the vector of the kinetic moment of the system onto the β axis, perpendicular to the orbital plane. These motions can be called *collinear relative equilibria*.

The method of parametric representation of relative equilibria for a satellite-gyrostat described in [5, 6] can also be used for this class of motions.

We will consider Eqs (1) as a linear system in β and γ, taking the rest of the parameters to be fixed. The solution of this system can be written in the form

$$\frac{\beta_i}{\sigma^*-I_i}=\frac{\gamma_i}{\lambda}=\frac{\dot\psi^{-1}K_i}{3\lambda^2+(\nu-I_i)(\sigma^*-I_i)}$$

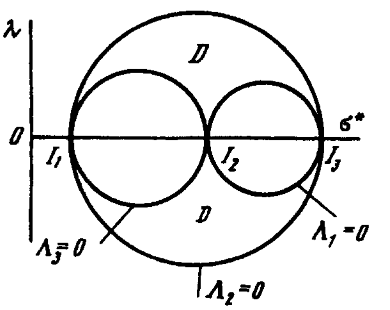

Fig. 2.

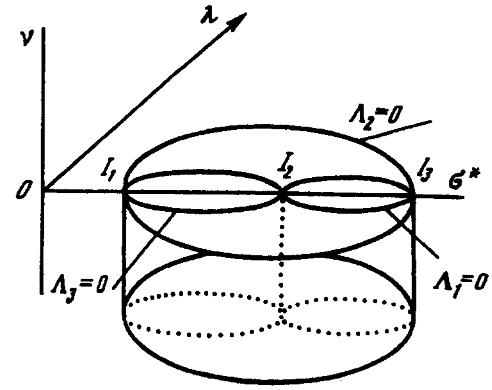

Fig. 3.

Substituting this solution into the orthonormality conditions for **β** and **γ**, we obtain a linear system of equations in K^2_1, K^2_2, K^2_3. Solving this system and substtuting the expressions for K^2_i into the relations for **β** and γ, we have

$$\gamma_1^2=\frac{(l_3-l_2)(\lambda^2+(\sigma^*-l_2)(\sigma^*-l_3))}{(l_2-l_3)(l_3-l_1)(l_1-l_2)},\quad \beta_1^2=\frac{(\sigma^*-l_1)^2}{\lambda^2}\gamma_1^2 (123) \tag{3}$$

Expressions that have not been written out explicitly are obtained by cyclical permutation of the indices.

Hence, the complete set of relative equilibria and the KMR which ensures them turns out to be the parametricized quantities (λ, σ*, ν). The expressions for these quantities, defining the orientation of the gyrostat, are independent of ν, a parameter which affects only the expressions for the KMR. The parameters (λ, σ*), however, are not arbitrary.

Suppose, for example, that

$$l_1<l_2<l_3 \tag{4}$$

The conditions under which the expressions on the right-hand sides of (3) are non-negative are

$$\Lambda_1=\lambda^2+(\sigma^*-l_2)(\sigma^*-l_3)\geq 0,\ \Lambda_2=\lambda^2+(\sigma^*-l_3)(\sigma^*-l_1)\leq 0,$$
$$\Lambda_3=\lambda^2+(\sigma^*-l_1)(\sigma^*-l_2)\geq 0 \tag{5}$$

These conditions are satisfied in a cylindrical region D of (λ, σ*, ν) space, bounded by the cylindrical surface $\{\Lambda_1 = 0, \Lambda_2 = 0, \Lambda_3 = 0\} \times R^1(\nu)$. The circular sections of D by the plane ν = 0, shown in Fig. 2, are similar to the Mohr circles of elasticity theory. The cylinders might be as shown in Fig. 3. Similar figures for a satellite-gyrostat have been found and investigated in [5, 6].

The straight lines λ = 0, $\sigma^* = l_i$, which form part of the boundary of D and touch the three cylinders taken in pairs, correspond to the families of relative equilibria

$$\gamma_1=1,\ \gamma_2=\gamma_3=\beta_1=0,\ \beta_2=\sin\theta,\ \beta_3=\cos\theta\ (123)$$

For the relative equilibria corresponding to the first index of cyclical permutation, the first axis of inertia indicates the attractive centre, and the gyrostat is rotated about this axis through an angle θ. The axis of the rotor is orthogonal to the vector γ and lies in the plane Cx_2x_3. The other relative equilibria can be interpreted in the same way. Relative equilibria of this class are similar to those of a satellite-gyrostat [4, 7].

The points of the cylinders $\Lambda_i = 0$ (i = 1, 2, 3) correspond to the relative equilibria

$$\gamma_1=\cos\theta,\ \gamma_2=-\sin\theta,\ \gamma_3=\beta_3=0,\ \beta_1=\sin\theta,\ \beta_2=\cos\theta\ (123)$$
$$\lambda=(l_2-l_3)\sin\theta\cos\theta,\quad \sigma^*=l_1\cos^2\theta+l_2\sin^2\theta$$
$$K_1\dot\psi^{-1}=(\nu-l_1+3(l_1-l_2)\cos^2\theta)\sin\theta$$
$$K_2\dot\psi^{-1}=(\nu-l_2+3(l_2-l_1)\sin^2\theta)\cos\theta,\ K_3=0$$

For the relative equilibria corresponding to the first index of cyclical permutation, the third axis of inertia is in the direction of the tangent to the orbit, defined by the vector α, the plane Cx_1x_2 coincides with the plane (**β**, γ), and the axis Cx_2 makes an angle θ with the vector **β**. The axis of the rotor lies in the plane Cx_1x_2, that is, it is orthogonal to the vector $\boldsymbol{\alpha}$. The other relative equilibria can be interpreted in the same way. The relative equilibria of this class are similar to those of the satellite-gyrostat investigated in [4, 7].

To conclude, it should be noted that Mohr circles arise in theoretical mechanics not only in connection with the dynamics of orbital system with gyrostats, but also in the theory of oscillations of systems with two degrees of freedom [8]. They also appear in the dynamics of solids with a fixed point [9].

A. Burov's research was supported financially partly by the Austrian Research Foundation (FWF) under Lisa Meitner grant M00282TEC, and partly by the International Association for the Promotion of Cooperation with Scientists from the Independent States of the Former Soviet Union (93-1621-ext), the Russian Foundation for Basic Research (96-01-00261, 96-15-96051), and the INTEGRATION project.

REFERENCES

1. MOCCIA, A ., VETRELLA, S. and GROSSI, M., Attitude dynamics and control of a vertical interferometric radar tethered altimeter. *J. Guid. Control Dyn.*, 1993, **16**, 2, 264–269.
2. HUGHES, P. C., *Spacecraft Attitude Dynamics*. John Wiley, New York, 1986.
3. BOUROV, A. A. and TRÖGER, H., On relative equilibria of a tethered gyrostat in a central Newtonian field. *ZAMM*, 1997, **77**, Suppl. 1, 53–54.
4. STEPANOV, S. Ya., On the set of steady motions of a satellite-gyrostat in a central Newtonian field of forces and the stability of their motions. *Prikl. Mat. Mekh.*, 1969, **33**, 4, 737–744.
5. RUBANOVSKII, V. N. and SAMSONOV, V . A., *Examples and Problems of the Stability Motion*. Nauka, Moscow, 1988.
6. RUBANOVSKII, V. N., The relative equilibria of a satellite-gyrostat, their branchings and stability. *Prikl. Mat. Mekh.*, 1988, **52**, 6, 909–914.
7. RUMYANTSEV, V. V., The Stability of the Steady Motion of Satellites. *Vychisl. Tsentr Akad. Nauk SSSR*, Moscow, 1967.
8. den HARTOG, J. P., *Mechanical Vibrations*. McGraw-Hill, New York, 1956.
9. RIMROTT, F. P. J. and SAITOV, A., Attitude diagrams for the descriptions of the attitude drift of torquefree asymmetric gyros. In *GAMM Tagung Programm. Prag Karls-Universitaet*, 1966, 46.

Translated by R.L.